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Double-pulse cotangential transfers between coplanar elliptic orbits[☆]

V.G. Adamyan[*], L.V. Adamyan, G.M. Zaimtsyan[✠], L.T. Manandyan

Gyumri, Armenia

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ABSTRACT

The geometric characteristics of double-impulse cotangential transfers between coplanar elliptic orbits, which are used to investigate of such transfers, are given. Each argument is accompanied by the development of a corresponding geometric algorithm which illuminates the mechanical problem from a geometric point of view, imparting the clarity to it which is characteristic of a geometric concept. A general method of investigation is developed based on a comparison of the behaviour of a cotangential transfer with an excentre of the transfer orbit which is joined to the excentres of the given elliptic orbits (an excentre is a circle constructed on the major axis of the ellipse which is its diameter). The cotangential transfer trajectory parameters and the values of the velocity pulses controlling the motion of the spacecraft during the transfer are determined in explicit form and depend on the parameters of the specified orbits and the true anomaly of the point of application of the first velocity pulse.

Problems of designing interorbital space flights[1–4] arise when carrying out the optimum transfer of a spacecraft from one orbit to another more suitable orbit. However, selecting the requisite transfer orbits between the coplanar orbits from the immense set is a difficult problem. It has been established that the manoeuvring of a spacecraft with minimum fuel expenditure is equivalent to manoeuvring with minimum overall changes in the orbital velocity.[1,5,6] The mathematical problem arising here is to determine in advance precisely which velocity pulses have to be imparted to the spacecraft and when this should be done in order that its orbit changes in the required manner. However, the problem of finding the flight trajectory between coplanar orbits for which the cost of the characteristic velocity is a minimum has still not been solved in general form.[1,7]

In the overwhelming majority of papers on celestial mechanics, the investigation of interorbital flights begins with a detailed treatment of the problem of the flight between two circular orbits. The particular form of such transfers in the shape of an ellipse touching both orbits was proposed by Hohmann in 1925. Analytical and numerical methods have been developed for other types of transfers, some of which are based on the variational calculus and the theory of approximate computations. Another, quite extensive part combines methods which have been specially developed using the theory of optimal control. The application of variational methods to optimization problems leads to two-point boundary value problems with all the difficulties invovled in solving them. When solving problems by these methods, the orbits considered are replaced by a set of points covering them discretely, and possible flight trajectories from the set of points of the initial orbit to the set of points of the final orbit are analysed. The optimal transfer orbit (OTO) is chosen by comparing the different versions of the flight considered from the point of view of the overall energy expenditure.[2]

However, the selected version may turn out to be considerably worse than optimal. The reason for this is concealed in the fact that it is impossible to solve problems involving the exact and complete description of the trajectory of the optimal transfer by the variational method since it is even impossible to consider all the transfer orbits and to compare them with one another with respect to fuel consumption, and it is necessary to restrict the treatment to just a small number of randomly selected versions. Attempts to determine the absolutely optimal flight trajectory between specified coplanar orbits by an approximate method are therefore not promising.

Numerical analysis of the research carried out shows that the OTO at the points of application of the pulses comes into contact with the specified orbits[2] and that the number of such cotangential transfers is infinite. The opinion[5] that a cotangential transfer is very close to optimal for any initial point of the transfer is erroneous since these transfers differ in the amount of energy consumed.

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* Corresponding author.
E-mail addresses: roman@gyumri.am, vgadamyan@mail.ru (V.G. Adamyan).

✠ Deceased.

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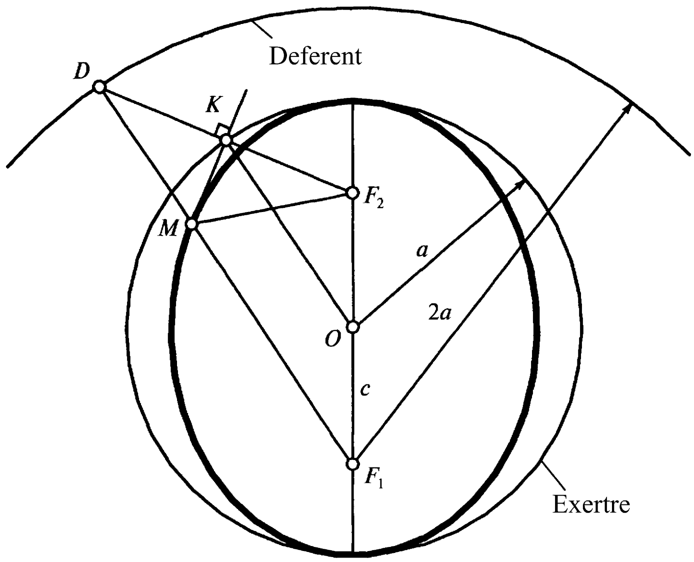


Fig. 1.

In spite of the cotangentialness of the OTO in existing papers, on the one hand, a set of transfers via intersecting transfer orbits which cannot claim to be OTOs is considered and, on the other hand, an infinite set of touching transfer orbits, each of which could turn out to be the required OTO, is dropped out of the treatment. Attempts to construct the exact OTO among coplanar orbits of general form, based on known methods of celestial mechanics, have not led to the proper result. The main reason for this failure is the fact that the inspection of the transfer orbits is carried out using both transfer points simultaneously. Moreover, there is no algorithm for choosing the OTO from all possible transfer orbits. It can be concluded that these difficulties can be removed if the OTO is considered in the system of all cotangential transfers between the specified orbits.

Since a unique cotangential flight orbit passes through each point of the initial orbit, the problem of finding the optimal solution reduces to finding the initial point of the transfer for which the corresponding cotangential transfer would be the required optimal flight orbit. It follows from this that, for a cotangential transfer, only the initial transfer point is independent and the point of application of the second pulse must not be specified as an initial condition since it can be determined by construction as a function of the point of application of the first pulse. No attention has been paid to this fact before.

A well known property of an ellipse and an Euclidean algorithm are the foundation of the development of a geometric algorithm which reproduces a number of functions analogous to the motion of bodies in the case of a cotangential transfer between elliptic orbits and enables one to establish the relation between the parameters of these orbits and the flight orbit.

Consider an ellipse with foci F_1 and F_2 and a major axis of length $2a$ (Fig. 1). On the continuation of the radius vector F_1M, we mark off a segment $MD = MF_2$. Then, $F_1D = 2a$. We join point D to the second focus F_2 and draw the height MK in the isosceles triangle F_2MD. It is obvious that MK will both be the bisectrix of the angle M and the median, that is, the line MK will be a tangent to the ellipse at the point M and the equality $DK = KF_2$ holds. We now join point K to the centre O of the ellipse. Since the segment KO is the middle line of the triangle F_1DF_2, we have $KO = F_1D/2 = a$.

Since the sum of the distances from the foci to any point of an ellipse is constant, it is obvious that the points D and K will move when the point M moves. The point D will move along a circle with centre F_1 and radius $2a$ (along the deferent according to the terminology of ancient astronomers) and the point K moves along a circle with centre O and radius a (along the excentre).

Before investigating problems associated with the construction and study of a cotangential transfer between coplanar orbits, we will consider the relation between the flight orbit and the separate specified orbits. It is obvious that, in order to understand this relation, it is necessary to investigate the family of trajectories emerging from a common point with the same direction of the initial velocity. It is well known that subfamilies of ellipses, hyperbolae and a parabola[8] are included in this family of trajectories. These curves, by definition, have a common focus, a common point and tangent to this point.

The aim of this paper is to find a second common property of the above mentioned curves which is important in the subsequent investigations.

The force centre F_1 and the initial position M_0 of the second point mass, which moves under the action of the attraction of the force centre F_1 are shown in Fig. 2. Suppose the initial velocity is directed at an angle α_0 to the radius vector r_0. The straight line *f*, which is the geometric locus of the foci F_2 corresponding to different values of the initial velocity V_0, is drawn through the point M_0. The straight line *f* makes an angle α_0 with the specified tangent since this tangent makes equal angles with the radius vectors M_0F_1 and M_0F_2. The position of the focus F_2 on the line *f* is uniquely defined by the value of modulus of the initial velocity V_0. The middle O of the interfocal distance F_1F_2 is the centre of the current trajectory of the family of trajectories considered with the same direction of the initial velocity. We draw a straight line parallel to the line *f* through the point O which passes through the middle O_0 of the radius vector F_1M_0 and intersects the direction of the initial velocity V_0 at the point S. From the isosceles triangle M_0O_0S with angles near the base $\angle O_0M_0S = \angle O_0SM_0 = \alpha_0$, we find that $M_0O_0 = O_0S = r_0/2$.

According to the definition of an ellipse, we have

$$F_1M_0 + F_2M_0 = 2a$$

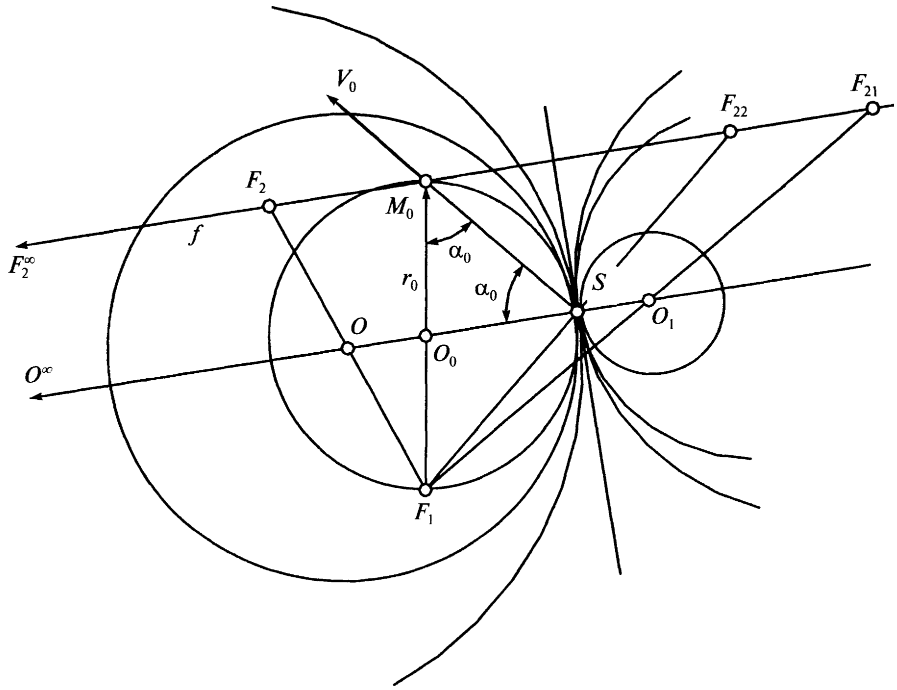


Fig. 2.

Since the segment OO_0 is the middle line of the triangle $F_1M_0F_2$, the semi-major axis a of the ellipse can be determined by the equality

$$F_1O_0 + O_0O = a$$

from which, when account is taken of the equality $F_1O_0 = O_0S = r_0/2$, we obtain that the segment OS gives the value of the semi-major axis a of the trajectory for all values of the initial velocity V_0.

This means that all the circles of the excentres of a trajectory of the family investigated pass through the point S on the line of their centres. These circles form a parabolic bundle of circles with centre S. If the initial velocity V_0 is equal to zero, we obtain an elliptic-type rectilinear motion along the line M_0F_1. The circle with centre O_0 and radius O_0S passes through the points M_0 and F_1 and the angle F_1SM_0 will be a right angle, since it is based on the diameter F_1M_0 of the excentre circle with zero initial velocity. If the centre O departs to infinity along the line $SO(V_0 = \sqrt{2g_0r_0})$, the excentre circle decomposes into two lines, one of which is an ideal line in the plane of the motion while the second is perpendicular to the line of the centres OS and passes through the point S. In this case, we obtain a parabolic trajectory.

When $V_0 > \sqrt{2g_0r_0}$, we have a subfamily of hyperbolic trajectories, the centre of which approaches the point S from the right-hand side as the initial velocity increases.

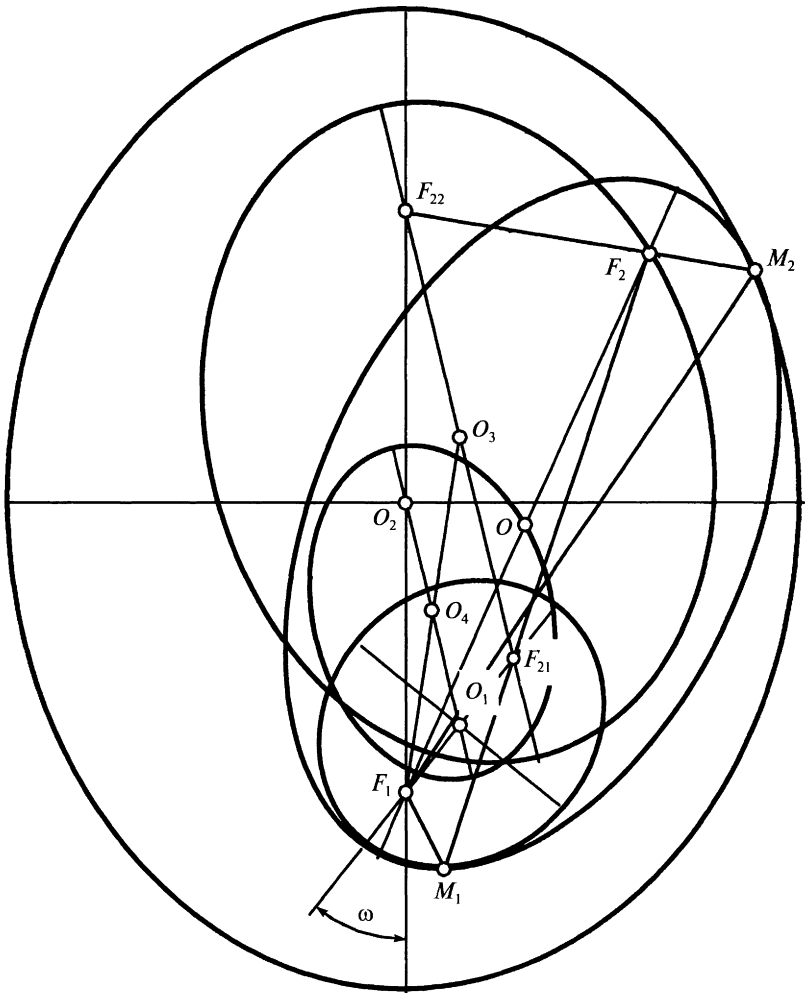
We now consider the problem of using a certain additional property of the family of trajectories with the same direction of the initial velocity to construct cotangential transfers. Since a cotangential transfer orbit touches the initial and the final orbits, it follows from what has been described above that, in this case, the excentre of the transfer orbit will simultaneously touch the excentres of the specified orbits, that is, their coupling holds.

Suppose the centre of the attractive forces F_1 and the elliptic orbits around the power centre F_1 are given (Fig. 3). The parameters a_1c_1 and a_2c_2 of these orbits and the angle ω between their major axes are known. Consider a cotangential transfer when the transfer orbit touches the initial orbit at the point M_1, and the final orbit at the point M_2. The second focus F_2 of the flight orbit is the point of intersection of the local lines M_1F_{21} and M_1F_{22}, where F_{21} and F_{22} are the second foci of the specified elliptic orbits.

Since the centre O of the transfer orbit is the middle of the interfocal distance F_1F_2, the geometric locus of the centres of all the cotangential transfers can be considered as a figure which is centrally-similar to a focal ellipse. Here, the centre of similarity is the attracting centre F_1 and the coefficient of similarity is equal to 1/2. It can then be concluded that, in the case of a cotangential orbital transfer between non-intersecting elliptic orbits, the geometric locus of the centre of the transfer orbits is an ellipse (the central curve), the foci of which coincide with the centres of the specified orbits and the semi-major axis is equal to half the difference between the semi-major axes of these orbits.

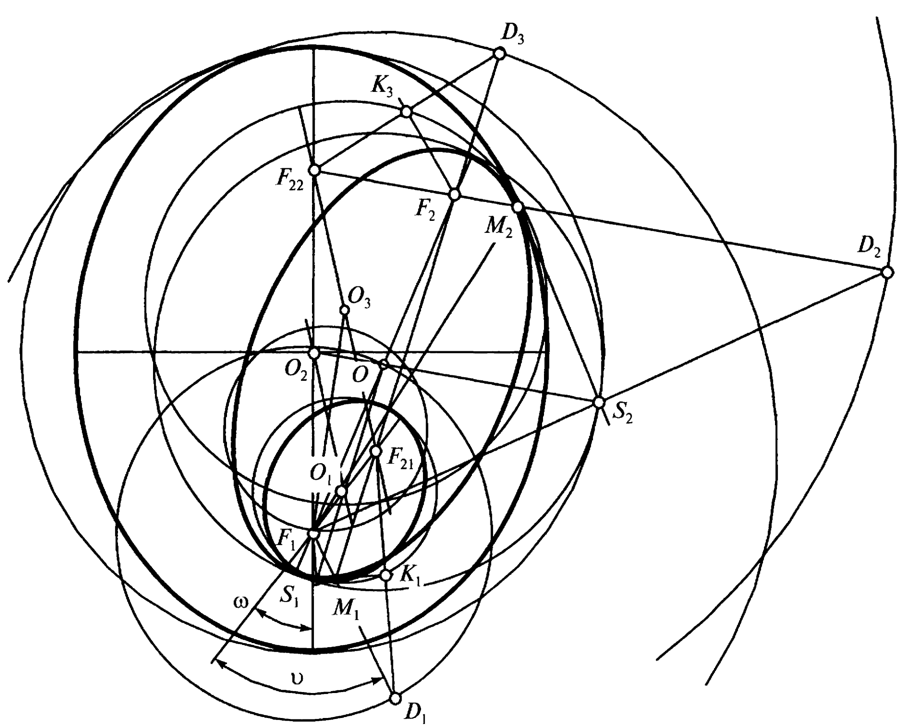
The above mentioned properties of cotangential transfers play a considerable large role in the subsequent investigations. We shall show that the problem of designing a cotangential transfer with a pair of compasses and a ruler, which provides a method for obtaining an analytical solution, is solvable using these properties.

With the aim of initially constructing cotangential transfers by the geometric route, we will develop the relations between the start and the finish of such transfers. Suppose the attracting centre F_1 and the elliptic orbits (a_1c_1) and (a_2c_2) around it as well as the angle ω between the major axes of the orbits are given. The points F_{21} and F_{22} are their foci, and O_1 and O_2 are their centres (Fig. 4). The centre of the focal ellipse is the middle O_3 of the interfocal distance $F_{21}F_{22}$. It is now necessary to draw two circles for each of the above mentioned ellipses, one of which with the centre of the ellipse being considered and a radius equal to its semi-major axis, and a second with its centre



F_{22}
F_2
M_2
O_3
O_2
O
O_4
F_{21}
O_1
F_1
M_1
ω

Fig. 3.



D_3
K_3
F_{22}
F_2
M_2
D_2
O_3
O_2
O
S_2
F_{21}
O_1
F_1
K_1
ω
S_1
M_1
υ
D_1

Fig. 4.

at one of the foci of this ellipse and with the radius of its major axis. This means that it is necessary to draw six circles with the parameters

$$(O_1, a_1), \quad (F_1, 2a_1), \quad (O_2, a_2), \quad (F_{22}, 2a_2), \quad (O_3, a_2 - a_1), \quad (F_{21}, 2(a_2 - a_1))$$

We begin the subsequent constructions by choosing the initial point M_1 of the cotangential transfer. To do this, we draw a line F_1D_1 through the attracting centre F_1 at an angle υ measured from the radius vector of the perigee of the initial orbit and determine the point D_1 of intersection of this line with the circle $(F_1,2a_1)$. We then draw a line D_1F_{21} and, through the point K_1 of its intersection with the circle (O_1, a_1), we draw a second line perpendicular to the line D_1F_{21}. This second line, on intersecting the line F_1D_1, gives the required point M_1 in the initial orbit.

We now determine the position of the second focus F_2 of the flight orbit in the focal ellipse. To do this, we join the point M_1 of the initial orbit with its second focus F_{21} and construct the point D_3 of intersection of the line M_1F_{21} with the circle F_{21}, $(2(a_2-a_1))$. We draw a line D_3F_{22} through the point D_3 and, through the point K_3 of its intersection with the circle (O_3, a_2-a_1), we draw a second line perpendicular to the line $F_{22}D_3$. This second line, on intersecting the line $F_{21}D_3$ gives the required focus F_2 of the flight orbit.

The point M_2 where the flight orbit touches the final orbit is determined as follows. We first join the second focus F_{22} of the final orbit with the second focus of the flight orbit by a straight line and determine its point of intersection D_2 with the circle $(F_{22}, 2a_2)$. We next join the point D_2 to the attracting centre F_1 by a straight line and draw a second line, which is perpendicular to the line (O_2,a_2), through its point of intersection S_2 with the circle F_1D_2. This second line, on intersecting the line $F_{22}D_3$, determines the required point M_2. We draw the lines F_1F_2 and O_2S_2 which, on intersecting, determine the required centre O of the flight orbit and the magnitude of its semi-major axis $a = OS_2$. We now join the centre O_1 of the initial orbit with the centre O of the flight orbit and determine its point of intersection with the initial excentre at the point of contact S_1 of the circles (O_1,a_1) and $(O, a_1 = OS_1 = OS_2)$.

Hence, using a pair of compasses and a ruler, the required orbit of cotangential transfer and its points of contact with the specified orbits can be successfully constructed. All the cotangential transfers arise for all possible positions of the point M_1, that is, the position of the point M_1 plays the role of a geometric parameter. It is well known that, if the possibility of constructing the required figure using a pair of compasses and a ruler is proved, then the lengths of all the segments, appearing in the construction, can be expressed in terms of the lengths of the specified segments using a finite number of elementary actions and the operation of extracting a square root.[9] This provides grounds for hoping that the explicit dependences of the parameters of the transfer orbit on the parameters of the specified orbits can be obtained. Fixing the value of the true anomaly υ of the point M_1, we obtain a completely defined flight orbit. If. however, the parameter υ changes continuously, we obtain a family of such orbits, and it is possible to write expressions for the parameters of the flight orbit in the form of corresponding functions of the argument υ. This means that functions, which depend on the true anomaly υ of the point M_1, rather than the values, will now be the unknowns.

The preliminary calculations carried out obove show that, by choosing the true anomaly υ of the point of contact M_1 of the flight orbit with the initial orbit as the independent variable, complex relations for the parameters of the flight orbit are obtained. In order to overcome this difficulty, it was decided to introduce a new independent variable. Bearing in mind that the centre of the transfer orbit is also the centre of its excentre, the radius of which is equal to the semi-major axis of the required flight orbit, the angle φ, which determines the position of the point of contact S_1 of the excentres of the transfer and initial orbits, can be taken as the independent variable.

Suppose the centre of the attracting forces F_1 and the elliptic orbits around the force centre F_1 are given (Fig. 5). The parameters (a_1,c_1) and (a_2,c_2) of these orbits and the angle ω between their major axes are known. The points F_{21} and F_{22} are the second foci of the specified orbits and O_1 and O_2 are their centres. The centre of the central curve is the middle O_3 of the segment O_1O_2. We now draw circles with the parameters

$$(O_1,a_1), \quad (F_{21},2a_1), \quad (O_2,a_2), \quad (F_{22},2a_2), \quad (O_3,(a_2 - a_1)/2), \quad (O_1,(a_2 - a_1))$$

We begin the further construction by choosing of the point S_1 on the excentre of the initial orbit. The angle φ of rotation of the radius O_1S_1 of the initial excentre is measured from the line, parallel to the F_1F_{22} axis, of the final elliptic orbit and passes through the centre O_1 of the initial elliptic orbit. We now determine the point of the intersection D_1 of the line F_1S_1 with the circle $(F_{21},2a_1)$. In order to construct the starting point of transfer M_1, we draw the line D_1F_{21} and then a line perpendicular to the line F_1D_1 through the point S_1. This last line, on intersecting the line $F_{21}D_1$, gives the required point M_1 in the initial orbit (Fig. 4).

We now determine of the position of the centre O of the flight orbit. To do this, we first construct the point of intersection D_3 of the line O_1S_1 with the circle (O_1,a_1-a_1). We draw the line D_3O_2 and, through its point of intersection K_3 with the circle $(O_3, (a_2-a_1)/2)$, a second line perpendicular to the line D_3O_2. This second line, on intersecting the line O_1D_3, gives the required centre O of the excentre of the flight orbit and its radius $a = 0S_1$. The lines of the centres O_2O, on intersecting the excentre of the final orbit, determine the point S_2 where it touches the excentre of the flight orbit.

The point M_2, where the flight orbit touches the final orbit, is determined in the following way. We first draw the line F_1S_2, determine its point of intersection D_2 with the circle $(F_{22},2a_2)$ and this point is joined by a straight line to the second focus F_{22} of the second orbit. A line, which is perpendicular to the line F_1D_2, is drawn through the point S_2 which, on intersecting the line $F_{22}D_2$, determines the required point M_2. The second focus of the flight orbit coincides with the point of intersection of the lines $F_{22}D_2$ and F_1O.

It should be pointed out that the specified orbits and flight orbit, which are shown in Fig. 4, are not presented in the algorithm which has been developed (Fig. 5). This is explained by the fact that the specified and transfer orbits are replaced by their excentres with the aim of simplifying the mathematical calculations. All the notation is the same in both figures.

In order to solve the problem analytically, we introduce a rectangular system of coordinates with origin at the attracting centre F_1 and with axis F_1y passing through the second focus F_{22} of the final orbit. We now determine the coordinates of the point D_3

$$X_{D_3} = X_{O_1} - O_1D_3\sin\varphi, \quad Y_{D_3} = Y_{O_1} + O_1D_3\cos\varphi$$

Taking the equalities

$$O_1D_3 = a_2 - a_1, \quad X_{O_1} = c_1\sin\omega, \quad Y_{O_1} = c_1\cos\omega$$

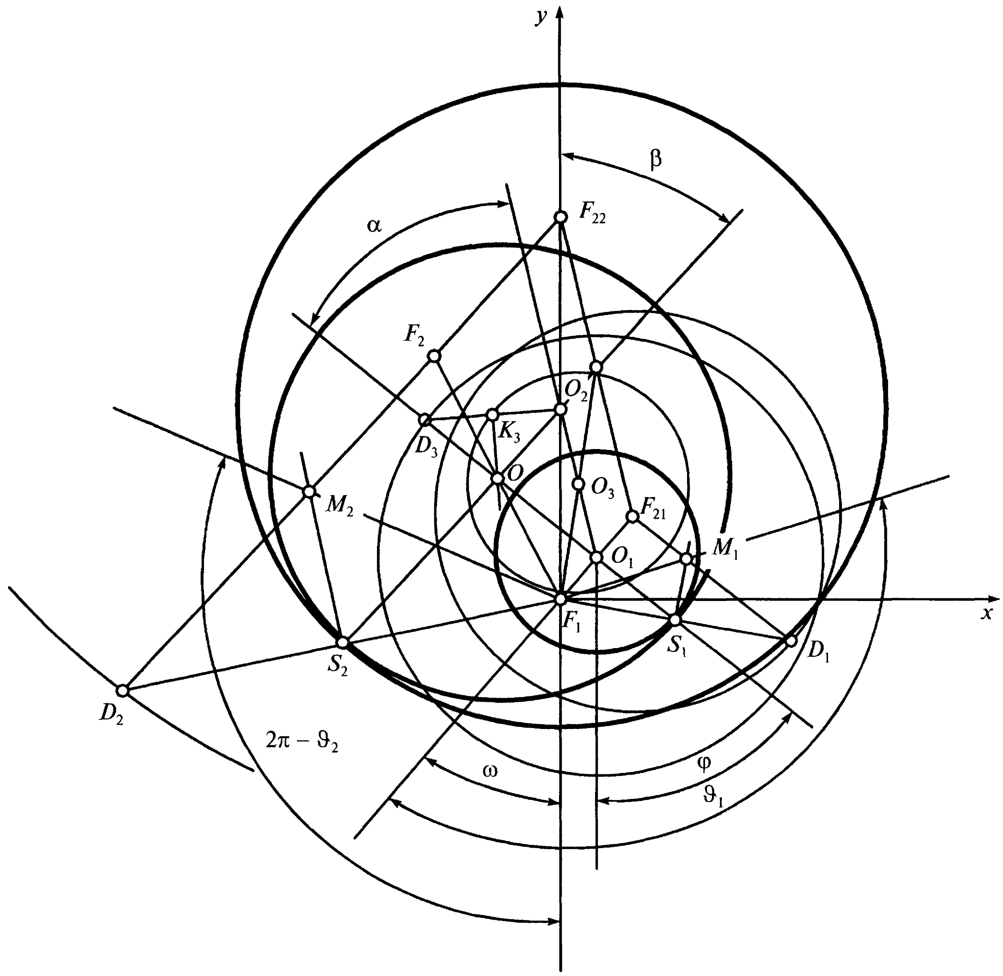


Fig. 5.

into account, after some reduction we obtain

$$X_{D3} = c_1 \sin\omega - (a_2 - a_1)\sin\varphi, \quad Y_{D_3} = c_1\cos\omega + (a_2 - a_1)\cos\varphi \tag{1}$$

The distance between the centres O_1 and O_2 is determined from the triangle $F_1O_1O_2$ using the cosine theorem

$$s = \sqrt{c_1^2 + c_2^2 - 2c_1c_2\cos\omega} \tag{2}$$

The distance between the points $O_2(0,c_2)$ and D_3 is determined using Pythagoras' theorem

$$O_2D_3 = \sqrt{X_{D_3}^2 + (c_2 - Y_{D3})^2}$$

from which, after taking account of equalities (1) and (2) and some reduction, we obtain

$$O_2D_3 = \sqrt{(a_2 - a_1)^2 + s^2 - (a_2 - a_1)\xi}, \quad \xi=2(c_2\cos\varphi - c_1\cos(\varphi + \omega)) \tag{3}$$

Using the cosine theorem, from the triangle $O_1O_2D_3$ with sides $O_1O_2 = s$, $O_1D_3 = a_2 - a_1$ and angle α between them we find a second expression for the length of the segment O_2D_3, which differs from expression (3) by the replacement of the quantity ξ by $2s\cos\alpha$. Equating these expressions, we obtain

$$\cos\alpha = \xi/2s \tag{4}$$

Using the cosine theorem and taking account of equalities (2) and (4), from the triangle O_1O_2O with the sides

$$O_1O_2 = s, \quad O_1O = O_1D_3 - OD_3 = a_2 - a_1 - OO_2$$

and an angle α between them, we find the length of the segment OO_2

$$OO_2 = [(a_2 - a_1)^2 + s^2 - (a_2 - a_1)\xi]/\eta, \quad \eta=2(a_2 - a_1) - \xi \tag{5}$$

Taking account of this equality, we determine the value of the radius of the excentre of the flight orbit

$$a = OS_1 = S_1D_3 - OD_3 = a_2 - OO_2 = [a_2^2 - a_1^2 - s^2 - a_1\xi]/\eta \tag{6}$$

as well as the distance between the centres O_1 and O of the initial and transfer orbits

$$O_1O = OS_1 - O_1S_1 = a - a_1 = \varsigma/\eta, \quad \varsigma = (a_2 - a_1)^2 - s^2 \tag{7}$$

and the coordinates of the point O, that is, of the centre of the flight orbit

$$\begin{aligned} X_0 &= X_{0_1} - O_1O\sin\varphi = c_1\sin\omega - \varsigma\sin\varphi/\eta \\ Y_0 &= Y_{0_1} + O_1O\cos\varphi = c_1\cos\omega + \varsigma\cos\varphi/\eta \end{aligned} \tag{8}$$

Using the cosine theorem, from the triangle F_1O_1O with sides $F_1O_1 = c_1$, $F_1O = c$ and angle $F_1O_1O = \pi - \angle F_1O_1S_1 = \pi - (\varphi + \omega)$ we find the square of the focal distance of the flight orbit

$$c^2 = c_1^2 + O_1O^2 + 2c_1O_1O\cos(\varphi + \omega) \tag{9}$$

from which, using equality (7), we find

$$c = \sqrt{c_1^2 + \varsigma^2/\eta^2 + 2c_1\varsigma\cos(\varphi + \omega)/\eta} \tag{10}$$

We will now determine the position of the apse of the flight trajectory. Since the point O is the centre of the flight orbit and the point F_1 is its focus, the slope of the line of the apses of the flight trajectory to the x axis is given by the formula $\operatorname{tg}\gamma = Y_0/X_0$, whence, after taking account of equalities (8) and some reduction, we obtain

$$\operatorname{tg}\gamma = (c_1\eta\cos\omega + \varsigma\cos\varphi)/(c_1\eta\sin\omega - \varsigma\sin\varphi) \tag{11}$$

Using the cosine theorem, from the triangle F_1OO_2 with sides $F_1O_2 = c$, $F_1O_2 = c_2$ and angle $F_1O_2O = \beta$ we find

$$c^2 = OO_2^2 + c_2^2 - 2OO_2c_2\cos\beta \tag{12}$$

Equating the right-hand sides of equalities (9) and (12), we obtain

$$\cos\beta = \frac{2(c_2 - c_1\cos\omega)(a_2 - a_1 + c_1\cos(\varphi + \omega)) - \left((a_2 - a_1)^2 + c_2^2 - c_1^2\right)\cos\varphi}{(a_2 - a_1)^2 + s^2 - (a_2 - a_1)\xi} \tag{13}$$

For the subsequent investigations, it is necessary to determine the value of the radius vectors r_1 and r_2 of the points M_1 and M_2 where the flight orbit touches the initial and the final orbits respectively. Using the cosine theorem, from the triangle $F_1F_{21}M_1$ with sides $F_1F_{21} = 2c_1$, $F_1M_1 = r_1$, $F_{21}M_1 = 2a_1 - r_1$ and angle $F_1F_{21}M_1 = \varphi + \omega$ we find

$$r_1^2 = 4c_1^2 + (2a_1 - r_1)^2 - 4c_1(2a_1 - r_1)\cos(\varphi + \omega)$$

whence, after simplifications, we obtain

$$r_1 = \frac{a_1^2 + c_1^2 - 2a_1c_1\cos(\varphi + \omega)}{a_1 - c_1\cos(\varphi + \omega)} \tag{14}$$

Similarly, from the triangle $F_1F_{22}M_2$ with sides $F_1F_{22} = 2c_2$, $F_1M_2 = r_2$, $F_{22}M_2 = 2a_2 - r_2$ and angle $F_1F_{22}M_2 = \beta$ we obtain

$$r_2 = \frac{a_2^2 + c_2^2 - 2a_2c_2\cos\beta}{a_2 - c_2\cos\beta} \tag{15}$$

The value of $\cos\beta$ is given by equality (13).

We will now determine the true anomalies ϑ_1 and ϑ_2 of the transition points M_1 and M_2, measured from the corresponding pericentres of the specified orbits. The values of the radius vectors r_1 and r_2 of the points M_1 and M_2 will be[8]

$$r_k = \frac{a_k^2 - c_k^2}{a_k + c_k\cos\vartheta_k}, \quad k = 1, 2 \tag{16}$$

From equalities (14) - (16), we obtain

$$\begin{aligned} \cos\vartheta_1 &= \frac{(a_1^2 + c_1^2)\cos(\varphi + \omega) - 2a_1c_1}{a_1^2 + c_1^2 - 2a_1c_1\cos(\varphi + \omega)}, \\ \cos\vartheta_2 &= \frac{(a_2^2 + c_2^2)\cos\beta - 2a_2c_2}{a_2^2 + c_2^2 - 2a_2c_2\cos\beta} \end{aligned} \tag{17}$$

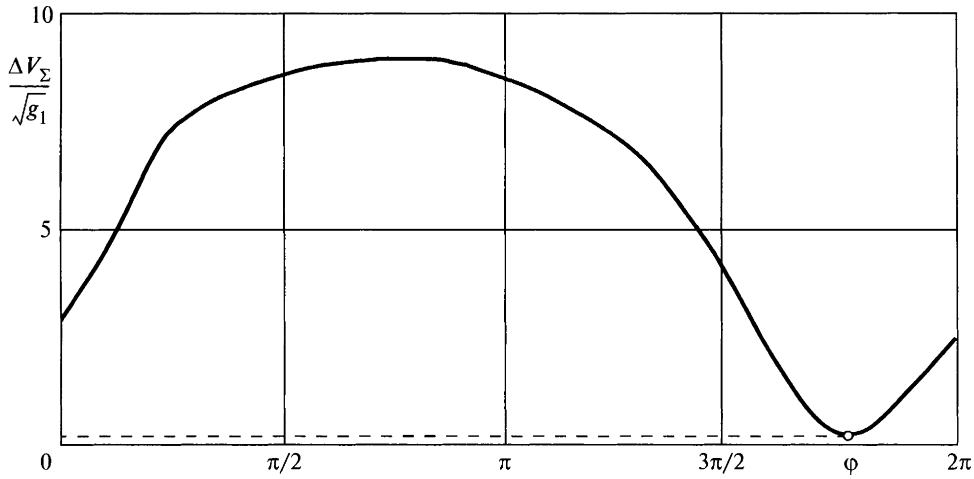


Fig. 6.

The angular distance in the flight trajectory is given by the equality

$$\vartheta = \vartheta_2 - \vartheta_1 + \omega$$

The true anomalies of the initial point M_1 and the final point M_2 are given by equalities (17).

The magnitudes of the semi-axes a_1,a_2 and a are given by the equalities[8]

$$a_k = \frac{g_k r_k^2}{2g_k r_k - V_k^2}, \quad a = \frac{g_k r_k^2}{2g_k r_k - V_{k1}^2}; \quad k = 1,2 \tag{18}$$

where g_1 and g_2 are the values of the Newtonian accelerations at the distances r_1 and r_2 respectively. From this, after some reduction, we obtain the moduli of the velocities V_1,$V_2 V_{11}$ and V_{21} of the spacecraft during its motion along the specified orbits and the flight orbit

$$V_k = \sqrt{\frac{g_k r_k (2a_k - r_k)}{a_k}}, \quad V_{k1} = \sqrt{\frac{g_k r_k (2a - r_k)}{a}}; \quad k = 1,2 \tag{19}$$

The magnitude of the velocity increment ΔV_1 can be determined, taking account of equalities (19), as the difference in the velocities V_{11} and V_1 of the spacecraft at the point M_1 which are required for motion along the flight orbit and the initial orbit:

$$\Delta V_1 = \sqrt{g_1 r_1}\left(\sqrt{\frac{2a - r_1}{a}} - \sqrt{\frac{2a_1 - r_1}{a_1}}\right) \tag{20}$$

Taking account of the equalities (19) and the condition $g_2 r_2 = g_1 r_1^2/r_2$, the value of the velocity increment ΔV_2 can be determined as the difference in the velocities V_2 and V_{21} of the spacecraft at the point M_2 required for the motion along the final and transition orbits:

$$\Delta V_2 = \sqrt{\frac{g_1 r_1^2}{r_2}}\left(\sqrt{\frac{2a_2 - r_2}{a_2}} - \sqrt{\frac{2a - r_2}{a}}\right) \tag{21}$$

The total velocity increment to perform the manoeuvre is given by the equality

$$\Delta V_\Sigma = \Delta V_1 + \Delta V_2 \tag{22}$$

Hence, the explicit dependences of the parameters of the cotangential transfer orbit and the moduli of the velocity increments on the parameters of the specified orbits have been found. The formulae obtained can be used for any transfer between coplanar elliptic and circular orbits.

It is clear from relations (20) - (22) that, for each point of contact S_1 of the excentres of the initial and transfer orbits, which is determined by the value of the angle of rotation φ of the radius $O_1 S_1$ of the initial excentre, determinate values of the velocity increments ΔV_1 and ΔV_2 and of the total velocity increment ΔV_Σ exist for a cotangential transfer. Note that the problem of minimizing the fuel consumption is equivalent to minimizing of the total velocity increment.

It is obvious that, in order to find the optimal flight orbit between the specified elliptic orbits, it is necessary to find the point M_1 on the initial orbit for which minimum energy consumption is required for the corresponding cotangential transfer.

The dependence of the total velocity increment on the angle φ for

$$a_1 = 20, \ c_1 = 11 \ (F_1 O_1), \ a_2 = 64, \ c_1 = 38 \ (F_1 O_2), \quad \omega = \pi/4.$$

is shown in Fig. 6. It can be seen that the total velocity increment has a minimum when $\varphi = 7\pi/4$.

The algorithm developed is suitable for every pair of coplanar elliptic orbits, depending on the parameters a_1,c_1,a_2,c_2 of these orbits and their mutual orientation, which is determined by the angle ω between their major axes. The specified orbits can touch, intersect or not have common points, and they can be coaxial or non-coaxial. In special cases, they can be circles. Here, there is no value for which, in the case of non-intersecting orbits, the local curve (the geometric locus of the second foci of the cotangential transfer orbits) is an ellipse and, in the case of intersecting orbits, a hyperbola. This is due to the fact that only their foci, that is, the second foci of the specified orbits and the circles of the excentre and deferent of the local curve, appear in the algorithm, which are the same for both cases.

The proposed algorithm for determining the parameters of the trajectory of the cotangential flight was modelled and tested using the Mathcad suite of programs. The results obtained confirmed the reliability of the algorithm and the formulae obtained.

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